CS 4850: Mathematical Foundations for the Information Age	March 12, 2010

Lecture 20

Lecturer: John Hopcroft | *Scribe: June Andrews*

Here we illustrate a proof method, treating a graph as an electrical network and considering the flow through the network. Edges become resistors and have directional current along them. For a given graph or network, we are interested in showing:

1. Increasing the resistance r_{xy} between nodes x and y, if anything can only increase the effective resistance of the network.
2. Thompson's Principle: that current distributes to minimize energy dissipation.

We will prove these through a series of small lemmas.

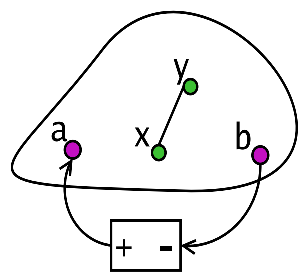

Figure 1: Treating a graph as an electrical network

For later reference, recall energy dissipation is defined as follows:

$$E = \frac{1}{2}\sum_{x,y} i_{xy}^2 r_{xy} = \frac{1}{2}\sum_{x,y} i_{xy}(v_x - v_y) \tag{1}$$

where the $\frac{1}{2}$ term comes from counting the current in the xy direction and yx direction in the double sum.
Now let us build from basics by defining a flow and proving lemmas for flows.

Definition 1 (Flow). *Flow is any function j on edges satisfying:*

1. $j_{xy} = -j_{yx}$*, ie flow is directional*
2. *Conservation of flow* $\sum_y j_{xy} = 0$*, except for the source and sink (a and b respectively and by convention for the rest of the lecture), where* $\sum_y j_{ay} = -1$ *and* $\sum_y j_{yb} = 1$
3. $j_{xy} = 0$ *if x and y are not connected*

Direct implications from the definition:

1. For non-source/sink nodes, $\sum_y j_{yx} = -\sum_y j_{xy} = 0$ by part 2 of the definition
2. Introducing the notation $j_x = \sum_y j_{xy}$ (a node x's net flow).

$j_a = -j_b$, where a is the source node and b is the sink node, since:

$$\begin{aligned}
j_a + j_b &= \sum_x j_x \text{ , by construction } j_x = 0 \text{ if } x \neq (a \text{ or } b) \\
&= \frac{1}{2} \sum_x \sum_y 2 j_{xy} \\
&= \frac{1}{2} \sum_{x,y} (j_{xy} - j_{yx}) \text{ , by flow definition part 1} \\
&= 0 \text{ , since every current and its negative will be added}
\end{aligned}$$

We now prove a useful lemma. Later, we will want to use the lemma with the voltage function.

Lemma 2. *For any function w on the nodes and any flow j:*

$$(w_a - w_b) j_a = \frac{1}{2} \sum_{x,y} (w_x - w_y) j_{xy}$$

Proof.

$$\begin{aligned}
\sum_{x,y} (w_x - w_y) j_{xy} &= \sum_{x,y} w_x j_{xy} - \sum_{x,y} w_y j_{xy} \\
&= \sum_x w_x \left(\sum_y j_{xy} \right) - \sum_y w_y \left(\sum_x j_{xy} \right) \\
&= \sum_x w_x j_x - \sum_y w_y (-j_y) \\
&= \sum_x w_x j_x + \sum_y w_y j_y \\
&= 2(w_a j_a + w_b j_b) \text{ , by construction } j_x = 0 \text{ if } x \neq (\text{a or b}) \\
&= 2(w_a - w_b) j_a \text{ , using } j_a = -j_b
\end{aligned}$$

□

Using the lemma and $V = IR$, we can now show the connection between the effective resistance of a network and the energy dissipated (use voltage(v) for function w and current(i) for flow j):

$$\begin{aligned}
i_a^2 r_{eff} &= i_a (v_a - v_b) \\
&= \frac{1}{2} \sum_{x,y} (v_x - v_y) i_{xy} \\
&= \frac{1}{2} \sum_{x,y} i_{xy}^2 r_{xy} \qquad (2)
\end{aligned}$$

On to Thompson's Principle!

Theorem 3 (Thompson Principle). *The current, i, in a network is the flow that minimizes energy.*

$$i = \arg\min_j \sum_{x,y} j_{xy}^2 r_{xy}$$

Proof. Let us set $j_a = i_a = 1$, and otherwise j is an arbitrary flow, and introduce the notation: $d_{xy} = j_{xy} - i_{xy}$.

We now calculate twice the energy dissipation for the arbitrary flow j:

$$\begin{aligned}\sum_{x,y} j_{xy}^2 r_{xy} &= \sum_{x,y}(d_{xy} + i_{xy})^2 r_{xy} \\ &= \sum_{x,y}\left[d_{xy}^2 r_{xy} + \sum_{x,y} 2d_{xy}i_{xy}r_{xy} + \sum_{x,y} i_{xy}^2 r_{xy}\right]\end{aligned}$$

Now suppose $\sum_{x,y} 2d_{xy}i_{xy}r_{xy} = 0$, then energy dissipated would have to be at least as big as the energy dissipation of the current. That would be a QED! We can in fact show this with the lemma if we get the sum into the proper form.

Note: the difference of 2 flows is a flow.

$$\begin{aligned}\sum_{x,y} 2d_{xy}i_{xy}r_{xy} &= 2\sum_{x,y} d_{xy}(v_x - v_y) \text{ , by } V = IR \\ &= 4(v_a - v_b)d_a \text{ , by the lemma} \\ &= 0\end{aligned}$$

□

We now have enough tools to prove the second main attraction:

Theorem 4. *if $r_{xy} \uparrow$ then, if anything, $r_{eff} \uparrow$.*

Proof. Let $\bar{r}$ be the resistance of the modified network, i be the current in the original network, and j be the current in the modified network.

Recall the relation of conservation of energy and effective resistance in equation 2:

$$i_a^2 r_{eff} = \frac{1}{2}\sum_{x,y} i_{xy}^2 r_{xy}$$

Now, without loss of generality, let $i_a^2 = j_a^2 = 1$

$$\begin{aligned}\bar{r}_{eff} &= \frac{1}{2}\sum_{x,y} j_{xy}^2 \bar{r}_{xy} \\ &\geq \frac{1}{2}\sum_{x,y} j_{xy}^2 r_{xy} \\ &\geq \frac{1}{2}\sum_{x,y} i_{xy}^2 r_{xy} \text{ , by Thompson's Principle} \\ &\geq r_{eff}\end{aligned}$$

□

Note: this is not a strict inequality, given the example in figure 2; also, by switching the definition of original/modified network, you show $r_{xy} \downarrow \implies$ if anything $r_{eff} \downarrow$.

⋆ Overall, we have shown that a lot of intuitive ideas can be easily shown with the proof through flow method.⋆

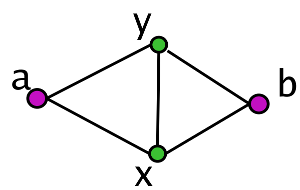

Figure 2: An example where changing r_{xy} would not change r_{eff}, since $i_{xy} = 0$

Introducing the topic of next time:

Imagine commute time from node u to v and back again:

- First cross $\vec{uv}$
- Perform a random walk until you traverse $\vec{uv}$ again

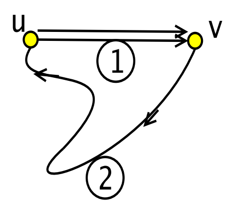

Figure 3: An Example illustrating a commute

Now, we know the expected time of the walk is $2m$ from class. Important: we assume random walks are memoryless.